

TOPOLOGICAL EQUIVALENCE FOR DISCONTINUOUS RANDOM DYNAMICAL SYSTEMS AND APPLICATIONS*

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ABSTRACT. After defining non-Gaussian Lévy processes for two-sided time, stochastic differential equations with such Lévy processes are considered. Solution paths for these stochastic differential equations have countable jump discontinuities in time. Topological equivalence (or conjugacy) for such an Itô stochastic differential equation and its transformed random differential equation is established. Consequently, a stochastic Hartman-Grobman theorem is proved for the linearization of the Itô stochastic differential equation. Furthermore, for Marcus stochastic differential equations, this topological equivalence is used to prove existence of global random attractors.

1. INTRODUCTION

Stochastic dynamical systems arise as mathematical models for complex phenomena under random fluctuations. They have been actively studied when the fluctuations are Gaussian [6, 11, 12]. Non-Gaussian random fluctuations are, however, more widely observed in various areas such as geophysics, biology, seismology, electrical engineering and finance [24, 15]. Lévy processes are a class of non-Gaussian processes whose sample paths are discontinuous in time. For a dynamical system driven by a Lévy process, almost all paths or orbits have countable jump discontinuities in time.

Discontinuous random dynamical systems or cocycles (Definition 2.1) generated by stochastic differential equations (SDEs) with Lévy processes have attracted attention more recently [14, 13, 18, 19].

In this paper, we consider topological equivalence between discontinuous cocycles, generated either by SDEs with Lévy processes or by related differential equations with random coefficients (i.e., random differential equations, or RDEs). Let us recall the definition on topological equivalence or conjugacy [2, 11].

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Definition 1.1. Two random dynamical systems φ and ψ on \mathbb{R}^d are called conjugate or topologically equivalent, if there exists a random homeomorphism $H : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that for all (ω, t) ,

$$\psi_t(\omega, \cdot) = H(\theta_t \omega, \cdot) \circ \varphi_t(\omega, \cdot) \circ H(\omega, \cdot)^{-1},$$

where $H(\omega, \cdot)^{-1}$ stands for the inverse mapping of $x \mapsto H(\omega, x)$. The homeomorphism H is called a cohomology of φ and ψ .

A cohomology of two random dynamical systems is a random coordinate transformation, which transforms the dynamical behavior for one of them into the dynamical behavior for the other. Moreover, it does not change the intrinsic asymptotic notions of these random dynamical systems, such as Lyapunov exponents and random attractors [11].

Conjugacy has been applied to study SDEs with Brownian motion in [11] and stochastic partial differential equations with Brownian motion in [5]. It has also been used to examine the stochastic flows for SDEs with Lévy processes in [18]. Consider the following SDE in \mathbb{R}^d : (Section 3.1)

$$\begin{cases} dX_t = a(X_t) dt + \sigma_i(X_t) dL_t^i, & t \geq 0, \\ X_0 = x, \end{cases} \quad (1)$$

and the RDE

$$\begin{cases} dY_t = \left(\frac{\partial \tilde{H}_t}{\partial x} \right)^{-1} (\omega, Y_t) a(\tilde{H}_t(\omega, Y_t)) dt, & t \geq 0, \\ Y_0 = x, \end{cases} \quad (2)$$

where the summation convention $\sum_i a_i b_i = a_i b_i$ is used, L_t^i 's are Lévy processes, and $\tilde{H}_t(\omega, x)$ is the solution of the equation

$$\tilde{H}_t(x) = x + \int_0^t \sigma_i(\tilde{H}_s(x)) dL_s^i. \quad (3)$$

In [18], under appropriate conditions, the first named author transformed SDE (1) to RDE (2) by $\tilde{H}_t(\omega, x)$. Then a homeomorphism stochastic flow property for Eq.(2) is proved in order to get the homeomorphism stochastic flow property for Eq.(1). Since $\tilde{H}_t(\omega, x)$ is generally different from $\tilde{H}_0(\theta_t \omega, x)$ for $t > 0$ and $\omega \in \Omega$, we need to modify SDE (3) to find a cohomology for topological equivalence.

In this paper, we transform SDEs with Lévy processes to RDEs by a cohomology. Because pathwise arguments for solutions of RDEs are more readily available than that of SDEs, dynamical problems such as linearization (i.e., Hartman-Grobman theorem) and random attractors of SDEs can be dealt with. Due to discontinuity in time for the solution paths of the SDEs with Lévy processes, the differentiation operation is conducted via Fubini theorem and integration by parts, instead of the Itô-ventzell formula. Additionally, for Marcus SDEs, we could only construct these cohomologies in special cases.

It is worth mentioning that a major source of discontinuous cocycles are solution mappings, after a perfection procedure [13], of stochastic differential equations with Lévy processes. To consider these cocycles with two-sided time \mathbb{R} , we need to define Lévy processes for two-sided time as well.

This paper is arranged as follows. In Section 2, we introduce discontinuous cocycles, and Lévy processes for two-sided time. In Section 3, we study conjugacy for Itô SDEs and RDEs and apply the result to prove a stochastic Hartman-Grobman theorem for SDEs with Lévy processes. Conjugacy for Marcus SDEs and RDEs, and existence of global

attractors are studied in Section 4. For readers' convenience, the Itô-Ventzell formula for càdlàg processes is placed in the Appendix, Section 5.

The following conventions will be used throughout the paper: C with or without indices will denote different positive constants (depending on the indices) whose values may change from one place to another.

2. PRELIMINARIES

In this section, we recall several basic concepts and results which will be needed throughout the paper.

2.1. Basic notations. The usual scalar product and norm (or length) in \mathbb{R}^d are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. Moreover, ∇ is the gradient of vector fields on \mathbb{R}^d , and $[\cdot, \cdot]$ is the Lie bracket operation on vector fields.

Denote by $\mathcal{C}_b^{m,\gamma}$ the set of functions $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + |x|} + \sum_{|\beta|=1}^m \sup_{x \in \mathbb{R}^d} |D^\beta f(x)| + \sum_{|\beta|=m} \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^\gamma} < \infty,$$

where $D^\beta f(x)$ stands for the β -order partial derivative of $f(x)$ for $|\beta| \leq m, m \in \mathbb{N}$ and $0 < \gamma < 1$.

2.2. Probability space. Let $D^*(\mathbb{R}, \mathbb{R}^d)$ be the set of all functions which are càdlàg (right continuous with left limit at each time) for $t \geq 0$ and càglàd (left continuous with right limit at each time) for $t \leq 0$, and take values in \mathbb{R}^d . We take canonical sample space $\Omega \triangleq D^*(\mathbb{R}, \mathbb{R}^d)$. It can be made a complete and separable metric space with endowed Skorohod metric ρ as follows ([3, 9]):

$$\rho(x, y) := \inf_{\lambda \in \Lambda} \left\{ \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| + \sum_{m=1}^{\infty} \frac{1}{2^m} \min \left\{ 1, \rho_m^\circ(x^m, y^m) \right\} \right\}$$

for all $x, y \in \Omega$, where $x^m(t) := g_m(t)x(t)$, $y^m(t) := g_m(t)y(t)$ with

$$g_m(t) := \begin{cases} 1, & \text{if } |t| \leq m, \\ m + 1 - |t|, & \text{if } m < |t| < m + 1, \\ 0, & \text{if } |t| \geq m + 1, \end{cases}$$

and

$$\rho_m^\circ(x, y) := \sup_{|t| \leq m} |x(t) - y(\lambda(t))|.$$

Here Λ denotes the set of strictly increasing and continuous functions from \mathbb{R} to \mathbb{R} . We identify a function $\omega(t)$ with a (canonical) sample path ω in the sample space Ω .

The Borel σ -algebra in the sample space Ω under the topology induced by the Skorohod metric ρ is denoted as \mathcal{F} . Note that $\mathcal{F} = \sigma(\omega(t), t \in \mathbb{R})$. Let \mathbb{P} be the unique probability measure which makes the canonical process a Lévy process for $t \in \mathbb{R}$ (see Definition 2.6 and 2.7 below). And we have the complete natural filtration $\mathcal{F}_s^t := \sigma(\omega(u) : s \leq u \leq t) \vee \mathcal{N}$ for $s \leq t$ with respect to \mathbb{P} .

2.3. Discontinuous random dynamical systems. ([2])

Define the Wiener shift

$$(\theta_t \omega)(\cdot) = \omega(t + \cdot) - \omega(t), \quad t \in \mathbb{R}, \quad \omega \in \Omega.$$

Then $\{\theta_t\}$ is a one-parameter group on Ω . In fact, Ω is invariant with respect to $\{\theta_t\}$, i.e.

$$\theta_t^{-1} \Omega = \Omega, \quad \text{for all } t \in \mathbb{R},$$

and \mathbb{P} is $\{\theta_t\}$ -invariant, i.e.

$$\mathbb{P}(\theta_t^{-1}(B)) = \mathbb{P}(B), \quad \text{for all } B \in \mathcal{F}, t \in \mathbb{R}.$$

Thus $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system (DS) and ergodic, i.e., all measurable $\{\theta_t\}$ -invariant sets have probability 0 or 1.

Definition 2.1. Let $(\mathbb{X}, \mathcal{B})$ be a measurable space. A mapping

$$\varphi : \mathbb{R} \times \Omega \times \mathbb{X} \mapsto \mathbb{X}, \quad (t, \omega, x) \mapsto \varphi_t(\omega, x)$$

with the following properties is called a measurable random dynamical system (RDS), or in short, a discontinuous cocycle:

- (i) *Measurability:* φ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}/\mathcal{B}$ -measurable,
- (ii) *Discontinuous cocycle (over θ) property:* $\varphi(t, \omega)$ is càdlàg for $t \geq 0$ and càglàd for $t \leq 0$ and furthermore

$$\varphi_0(\omega, \cdot) = id_{\mathbb{X}}, \quad \text{for all } \omega \in \Omega, \quad (4)$$

$$\varphi_{t+s}(\omega, \cdot) = \varphi_t(\theta_s \omega, \cdot) \circ \varphi_s(\omega, \cdot), \quad \text{for all } s, t \in \mathbb{R}, \quad \omega \in \Omega. \quad (5)$$

2.4. Random attractors. We recall the definition of a random attractor ([5]) and a theorem for its existence ([6]). Let φ_t be a discontinuous cocycle.

Definition 2.2. A random bounded set $B(\omega) \subset \mathbb{R}^d$ is called tempered if

$$\lim_{t \rightarrow +\infty} e^{-\beta t} d(B(\theta_{-t} \omega)) = 0, \quad \text{for any } \beta > 0,$$

where $d(A) = \sup_{x \in A} |x|$.

Definition 2.3. A random set $K(\omega)$ is said to be an absorbing set if for all tempered random bounded set $B(\omega) \subset \mathbb{R}^d$ there exists $t(\omega, B) > 0$ such that

$$\varphi_t(\theta_{-t} \omega, B(\theta_{-t} \omega)) \subset K(\omega), \quad \text{for all } t \geq t(\omega, B).$$

Definition 2.4. A random compact set $A(\omega) \subset \mathbb{R}^d$ is called a global random attractor of φ if for all $\omega \in \Omega$

$$\begin{aligned} \varphi_t(\omega, A(\omega)) &= A(\theta_t \omega), \quad \text{for all } t \geq 0, \\ \lim_{t \rightarrow +\infty} \text{dist}(\varphi_t(\theta_{-t} \omega, B(\theta_{-t} \omega)), A(\omega)) &= 0, \end{aligned}$$

for all tempered random bounded set $B(\omega) \subset \mathbb{R}^d$, where dist denotes the semi-Hausdorff distance

$$\text{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|.$$

Note that a random attractor is unique. The following existence theorem is from [6].

Theorem 2.5. If there exists a compact absorbing set $K(\omega)$, then there exists a random attractor of φ .

2.5. Lévy processes.

2.5.1. Lévy processes for $t \geq 0$.

Definition 2.6. A stochastic process $L = (L_t)_{t \geq 0}$ with $L_0 = 0$ a.s. is a d -dimensional Lévy process if

(i) L has independent increments; that is, $L_t - L_s$ is independent of $L_v - L_u$ if $(u, v) \cap (s, t) = \emptyset$;

(ii) L has stationary increments; that is, $L_t - L_s$ has the same distribution as $L_v - L_u$ if $t - s = v - u > 0$;

(iii) L_t is right continuous with left limit.

Its characteristic function is given by

$$\mathbb{E}(\exp\{i\langle z, L_t \rangle\}) = \exp\{t\Psi(z)\}, \quad z \in \mathbb{R}^d.$$

The function $\Psi : \mathbb{R}^d \rightarrow \mathbb{C}$ is called the characteristic exponent of the Lévy process L . By the Lévy-Khintchine formula, there exist a nonnegative-definite $d \times d$ matrix Q , $b \in \mathbb{R}^d$ and a measure ν on \mathbb{R}^d satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|u|^2 \wedge 1) \nu(du) < \infty, \quad (6)$$

such that

$$\begin{aligned} \Psi(z) &= -\frac{1}{2} \langle z, Qz \rangle + i \langle z, b \rangle \\ &\quad + \int_{\mathbb{R}^d} (e^{i\langle z, u \rangle} - 1 - i \langle z, u \rangle 1_{|u| \leq \delta}) \nu(du), \end{aligned} \quad (7)$$

where $\delta > 0$ is a constant. ν is called the Lévy measure.

Set $\kappa_t := L_t - L_{t-}$. Then κ defines a stationary $(\mathcal{F}_0^t)_{t \geq 0}$ -adapted Poisson point process with values in $\mathbb{R}^d \setminus \{0\}$ and the characteristic measure ν ([10]). Let $N_\kappa((0, t], du)$ be the counting measure of κ_t , i.e., for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$

$$N_\kappa((0, t], B) := \#\{0 < s \leq t : \kappa_s \in B\},$$

where $\#$ denotes the cardinality of a set. The compensator measure of N_κ is given by

$$\tilde{N}_\kappa((0, t], du) := N_\kappa((0, t], du) - t\nu(du).$$

The Lévy-Itô theorem states that there exist a d' -dimensional $(\mathcal{F}_0^t)_{t \geq 0}$ -Brownian motion W_t , $0 \leq d' \leq d$ and a $d \times d'$ matrix A such that L can be represented as

$$\begin{aligned} L_t &= bt + AW_t + \int_0^t \int_{|u| \leq \delta} u \tilde{N}_\kappa(ds, du) \\ &\quad + \int_0^t \int_{|u| > \delta} u N_\kappa(ds, du). \end{aligned} \quad (8)$$

2.5.2. Lévy processes for $t \leq 0$.

Definition 2.7. A stochastic process $L^- = (L_t^-)_{t \leq 0}$ with $L_0^- = 0$ a.s. is a d -dimensional Lévy process if

- (i) L^- has independent increments; that is, $L_t^- - L_s^-$ is independent of $L_v^- - L_u^-$ if $(v, u) \cap (t, s) = \emptyset$;
- (ii) L^- has stationary increments; that is, $L_t^- - L_s^-$ has the same distribution as $L_v^- - L_u^-$ if $t - s = v - u < 0$;
- (iii) L_t^- is left continuous with right limit.

Its characteristic function, characteristic exponent and the Lévy-Khintchine formula are the same as those for the Lévy process $L = (L_t)_{t \geq 0}$.

Set $\kappa_t^- := L_t^- - L_{t+}^-$. Then κ^- defines a stationary $(\mathcal{F}_t^0)_{t \leq 0}$ -adapted Poisson point process with values in $\mathbb{R}^d \setminus \{0\}$ and characteristic measure ν^- ([10]). Define

$$N_{\kappa^-}([t, 0), B) := \#\{t \leq s < 0 : \kappa_s^- \in B\},$$

for $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. The compensator measure of N_{κ^-} is given by

$$\tilde{N}_{\kappa^-}([t, 0), du) := N_{\kappa^-}([t, 0), du) + t\nu^-(du),$$

where ν^- is the Lévy measure for L^- . By the similar proof to that of the Lévy-Itô theorem in [20] we obtain that there exist $b^- \in \mathbb{R}^d$, a d'' -dimensional $(\mathcal{F}_t^0)_{t \leq 0}$ -Brownian motion W_t^- , $0 \leq d'' \leq d$ and a $d \times d''$ matrix A^- such that L^- can be represented as

$$\begin{aligned} L_t^- &= -b^-t - A^-W_t^- - \int_t^0 \int_{|u| \leq \delta} u \tilde{N}_{\kappa^-}(ds, du) \\ &\quad - \int_t^0 \int_{|u| > \delta} u N_{\kappa^-}(ds, du). \end{aligned} \tag{9}$$

A Lévy process with two-sided time, $t \in \mathbb{R}$, is defined to satisfy both Definitions 2.6 and 2.7.

3. CONJUGACY FOR ITÔ SDE AND RDE, AND LINEARIZATION OF SDES

In this section, we study conjugacy for Itô SDEs and RDEs and apply the result to study linearization of SDEs with Lévy processes.

3.1. Conjugacy for Itô SDEs and RDEs. Take a one-sided m -dimensional Lévy process

$$L_t = bt + AW_t + \int_0^t \int_{|u| \leq \delta} u \tilde{N}_{\kappa}(ds, du), \quad t \geq 0,$$

where W_t is a m' -dimensional $(\mathcal{F}_t^0)_{t \geq 0}$ -Brownian motion, $0 \leq m' \leq m$, $A = (a_{ij})$ is a $m \times m'$ matrix, $0 < \delta < 1$, and consider the following Itô SDE on \mathbb{R}^d for $y \in \mathbb{R}^d, \eta \in \mathbb{R}$:

$$\begin{cases} dY_t = \eta \sigma_i(Y_t) dL_t^i, & t \geq 0, \\ Y_0 = y, \end{cases} \tag{10}$$

where the differential is in the Itô sense ([10]).

Proposition 3.1. Assume that $\sigma_i(x) \in \mathcal{C}_b^{1,\gamma}$ for $i = 1, 2, \dots, m$. If

$$\begin{pmatrix} y \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} y \\ \eta \end{pmatrix} + \begin{pmatrix} \eta \sigma_i(y) u^i \\ 0 \end{pmatrix}$$

is homeomorphic for any $u \in \{u \in \mathbb{R}^m : |u| \leq \delta\}$ and the Jacobian matrix

$$I + \begin{pmatrix} \eta \frac{\partial(\sigma_i(y))^1}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^1}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^1}{\partial y_d} u^i & (\sigma_i(y))^1 u^i \\ \eta \frac{\partial(\sigma_i(y))^2}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^2}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^2}{\partial y_d} u^i & (\sigma_i(y))^2 u^i \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \eta \frac{\partial(\sigma_i(y))^d}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^d}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^d}{\partial y_d} u^i & (\sigma_i(y))^d u^i \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is invertible for any y, η a.e. and $u \in \{u \in \mathbb{R}^m : |u| \leq \delta\}$, then the solution $Y_t^{y,\eta}$ to (10) defines a stochastic flow of \mathcal{C}^1 -diffeomorphisms. Moreover, $Y_t^{y,\eta}$ is differentiable in η .

Proof. For any $\eta \in \mathbb{R}$, define $\eta_t := \eta$ and rewrite (10) as

$$\begin{aligned} \bar{Y}_t(\bar{y}) &= \bar{y} + \int_0^t F(\bar{Y}_s(\bar{y})) ds + \int_0^t G(\bar{Y}_s(\bar{y})) dW_s \\ &\quad + \int_0^t \int_{|u| \leq \delta} J(\bar{Y}_s(\bar{y}), u) \tilde{N}_\kappa(ds, du), \end{aligned} \quad (11)$$

where

$$\begin{aligned} \bar{Y}_t &= \begin{pmatrix} Y_t^{y,\eta} \\ \eta_t \end{pmatrix}, \quad \bar{y} = \begin{pmatrix} y \\ \eta \end{pmatrix}, \\ F(\bar{Y}_t) &= \begin{pmatrix} \eta \sigma_i(Y_t^{y,\eta}) b^i \\ 0 \end{pmatrix}, \quad J(\bar{Y}_t, u) = \begin{pmatrix} \eta \sigma_i(Y_t^{y,\eta}) u^i \\ 0 \end{pmatrix}, \\ G(\bar{Y}_t) &= \begin{pmatrix} \eta \sigma_i(Y_t^{y,\eta}) a^{i1} & \eta \sigma_i(Y_t^{y,\eta}) a^{i2} & \dots & \eta \sigma_i(Y_t^{y,\eta}) a^{im'} \\ 0 & 0 & \dots & 0 \end{pmatrix}. \end{aligned}$$

By Theorem 3.11 in [14], for almost all ω and $t \geq 0$, $\bar{y} \mapsto \bar{Y}_t(\bar{y})$ is a stochastic flow of local \mathcal{C}^1 -diffeomorphisms on $\mathbb{R}^d \times \mathbb{R}$.

Now we prove that these diffeomorphisms are global. For any $R > 0$, $|\bar{y}| \leq R$. Fix $T > 0$ and $p > d + 1$. To (11), by BDG inequality in [14, Theorem 2.11] and Hölder's inequality we have that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{Y}_t|^p \right) &\leq 4^{p-1} |\bar{y}|^p + 4^{p-1} T^{p-1} \mathbb{E} \left(\int_0^T |F(\bar{Y}_s(\bar{y}))|^p ds \right) \\ &\quad + 4^{p-1} C \mathbb{E} \left(\int_0^T \int_{|u| \leq \delta} |J(\bar{Y}_s(\bar{y}), u)|^p \nu(du) ds \right) \\ &\quad + 4^{p-1} C \mathbb{E} \left(\int_0^T \int_{|u| \leq \delta} |J(\bar{Y}_s(\bar{y}), u)|^2 \nu(du) ds \right)^{\frac{p}{2}} \\ &\quad + 4^{p-1} C \mathbb{E} \left(\int_0^T |G(\bar{Y}_s(\bar{y}))|^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

Noting that

$$|F(\bar{Y}_t)| \leq C(1 + |\bar{Y}_t|),$$

$$\begin{aligned} |G(\bar{Y}_t)| &\leq C(1 + |\bar{Y}_t|), \\ |J(\bar{Y}_t, u)| &\leq C(1 + |\bar{Y}_t|)u, \end{aligned}$$

we obtain that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{Y}_t|^p \right) &\leq 4^{p-1} |\bar{y}|^p + 4^{p-1} C \mathbb{E} \left(\int_0^T (1 + |\bar{Y}_s|^p) ds \right) \\ &\leq 4^{p-1} C (1 + |\bar{y}|^p) + 4^{p-1} C \mathbb{E} \left(\int_0^T \left(\sup_{0 \leq s \leq t} |\bar{Y}_s|^p \right) dt \right). \end{aligned}$$

Thus, Gronwall's inequality leads us to

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{Y}_t|^p \right) \leq 4^{p-1} C (1 + |\bar{y}|^p) e^{4^{p-1} CT}.$$

For $\bar{y}^1 = \begin{pmatrix} y^1 \\ \eta^1 \end{pmatrix}$ and $\bar{y}^2 = \begin{pmatrix} y^2 \\ \eta^2 \end{pmatrix}$, by the similar deduction to the above one, we can have that

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{Y}_t^1 - \bar{Y}_t^2|^p \right) \leq C |\bar{y}^1 - \bar{y}^2|^p.$$

By the Kolmogorov's continuity criterion in [7], it yields that for almost all $\omega \in \Omega$, $\bar{y} \mapsto \bar{Y}_t(\bar{y})$ is continuous on $\mathbb{R}^d \times \mathbb{R}$ for all $t \geq 0$. This completes the proof. \square

Take a two-sided m -dimensional Lévy process

$$\hat{L}_t = \begin{cases} bt + AW_t + \int_0^t \int_{|u| \leq \delta} u \tilde{N}_\kappa(ds, du), & t \geq 0, \\ -b^-t - A^-W_t^- - \int_t^0 \int_{|u| \leq \delta} u \tilde{N}_{\kappa^-}(ds, du), & t < 0. \end{cases}$$

We consider the following stochastic integral equations: for $x \in \mathbb{R}^d, \tau \in \mathbb{R}$,

$$h_t^{x, \tau} = x + e^{-\tau} \int_{-\infty}^t e^s \sigma_i(h_s^{x, \tau}) d\hat{L}_s^i, \quad t \in \mathbb{R}, \quad (12)$$

$$\tilde{h}_t^{x, \tau} = x + e^{-\tau} \int_0^t \sigma_i(\tilde{h}_s^{x, \tau}) d\tilde{L}_s^i, \quad t > 0, \quad (13)$$

where $\tilde{L}_t := \int_{-\infty}^{\frac{1}{2} \log 2t} e^s d\hat{L}_s$. So, $\tilde{h}_t^{x, \tau} = h_{\frac{1}{2} \log 2t}^{x, \tau}$.

By Definition 2.5.1, we know that \tilde{L}_t is a Lévy process. Thus, Proposition 3.1 implies that $x \mapsto \tilde{h}_t^{x, \tau}$ is an a.s. \mathcal{C}^1 -diffeomorphism of \mathbb{R}^d . Then $x \mapsto h_t^{x, \tau}$ is also a diffeomorphism of \mathbb{R}^d and $h_t^{x, \tau}$ is differentiable in τ . Set

$$H_t(\omega, x) := h_t^{x, \tau}|_{\tau=t}, \quad \Gamma_t(\omega, x) := \frac{\partial h_t^{x, \tau}}{\partial \tau}|_{\tau=t}, \quad t \in \mathbb{R}.$$

Proposition 3.2. *The following results hold:*

(i)

$$dH_t = \Gamma_t dt + \sigma_i(H_t) d\hat{L}_t^i,$$

(ii)

$$H_{s+t}(\omega, x) = H_t(\theta_s \omega, x), \quad \Gamma_{s+t}(\omega, x) = \Gamma_t(\theta_s \omega, x), \quad s, t \in \mathbb{R},$$

for a.s. ω .

Proof. Set

$$D(t, \tau) := \frac{\partial h_t^{x, \tau}}{\partial \tau}.$$

Then $D(t, t) = \Gamma_t$, and it satisfies the following equation

$$\begin{aligned} D(t, \tau) &= -e^{-\tau} \int_{-\infty}^t e^s \sigma_i(h_s^{x, \tau}) d\hat{L}_s^i \\ &\quad + e^{-\tau} \int_{-\infty}^t e^s \frac{\partial \sigma_i}{\partial x}(h_s^{x, \tau}) D(s, \tau) d\hat{L}_s^i. \end{aligned}$$

So, for $s \leq t$, $s, t \in \mathbb{R}$,

$$\begin{aligned} H_t &= x + e^{-t} \int_{-\infty}^t e^r \sigma_i(h_r^{x, t}) d\hat{L}_r^i \\ &= x + e^{-t} \int_{-\infty}^s e^r \sigma_i(h_r^{x, t}) d\hat{L}_r^i + e^{-t} \int_s^t e^r \sigma_i(h_r^{x, t}) d\hat{L}_r^i \\ &= x + e^{-t} \int_{-\infty}^s e^r \sigma_i(h_r^{x, t}) d\hat{L}_r^i + \int_s^t \sigma_i(h_r^{x, r}) d\hat{L}_r^i \\ &\quad - \int_s^t \left(\int_s^u e^{r-u} \sigma_i(h_r^{x, u}) d\hat{L}_r^i \right) du \\ &\quad + \int_s^t \left(\int_s^u e^{r-u} \frac{\partial \sigma_i}{\partial x}(h_r^{x, u}) D(r, u) d\hat{L}_r^i \right) du \\ &= x + \int_s^t \Gamma_u du + \int_s^t \sigma_i(H_r) d\hat{L}_r^i + e^{-t} \int_{-\infty}^s e^r \sigma_i(h_r^{x, t}) d\hat{L}_r^i \\ &\quad + \int_s^t e^{-u} \left(\int_{-\infty}^s e^r \sigma_i(h_r^{x, u}) d\hat{L}_r^i \right) du \\ &\quad - \int_s^t e^{-u} \left(\int_{-\infty}^s e^r \frac{\partial \sigma_i}{\partial x}(h_r^{x, u}) D(r, u) d\hat{L}_r^i \right) du, \end{aligned}$$

where we have used the following formula

$$\begin{aligned} e^{-t} \int_s^t e^r \sigma_i(h_r^{x, t}) d\hat{L}_r^i - \int_s^t \sigma_i(h_r^{x, r}) d\hat{L}_r^i &= \int_s^t \left(\int_s^u e^{r-u} \frac{\partial \sigma_i}{\partial x}(h_r^{x, u}) D(r, u) d\hat{L}_r^i \right) du \\ &\quad - \int_s^t \left(\int_s^u e^{r-u} \sigma_i(h_r^{x, u}) d\hat{L}_r^i \right) du. \end{aligned} \quad (14)$$

The proof of this formula is in Appendix.

Via integration by parts,

$$\begin{aligned} H_t &= x + \int_s^t \Gamma_u du + \int_s^t \sigma_i(H_r) d\hat{L}_r^i + e^{-s} \int_{-\infty}^s e^r \sigma_i(h_r^{x, s}) d\hat{L}_r^i \\ &= H_s + \int_s^t \Gamma_u du + \int_s^t \sigma_i(H_r) d\hat{L}_r^i. \end{aligned}$$

Thus,

$$dH_t = \Gamma_t dt + \sigma_i(H_t) d\hat{L}_t^i.$$

To (ii), for $s, t \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned}
H_{s+t}(\omega, x) &= x + e^{-(s+t)} \int_{-\infty}^{s+t} e^r \sigma_i(H_r(\omega, x)) d\hat{L}_r^i \\
&= x + e^{-(s+t)} \int_{-\infty}^t e^{s+u} \sigma_i(H_{s+u}(\omega, x)) d(\hat{L}_{s+u}^i - \hat{L}_s^i) \\
&= x + e^{-t} \int_{-\infty}^t e^u \sigma_i(H_{s+u}(\omega, x)) d\hat{L}_u^i(\theta_s \omega).
\end{aligned}$$

By the uniqueness of the solution to Eq.(12), we get

$$H_{s+t}(\omega, x) = H_t(\theta_s \omega, x)$$

for a.s. ω and $s, t \in \mathbb{R}$. By a similar perfection procedure to one in [13], we obtain the first result of (ii). The similar deduction to the above one admits us to have that $\Gamma_{s+t}(\omega, x) = \Gamma_t(\theta_s \omega, x)$. \square

By the above Proposition, we know that H_t and Γ_t are stationary processes.

Now we consider the SDE

$$dX_t = a(X_t)dt + \sigma_i(X_t)d\hat{L}_t^i, \quad t \in \mathbb{R}, \quad (15)$$

and the RDE

$$dY_t = \left(\frac{\partial}{\partial x} H_t \right)^{-1} (\omega, Y_t) \left[a(H_t(\omega, Y_t)) - \Gamma_t(\omega, Y_t) \right] dt, \quad t \in \mathbb{R}. \quad (16)$$

Proposition 3.3. Assume that $a(x) \in \mathcal{C}_b^{1,\gamma}$, $\sigma_i(x) \in \mathcal{C}_b^{1,\gamma}$ for $i = 1, 2, \dots, m$, the mapping

$$\begin{pmatrix} y \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} y \\ \eta \end{pmatrix} + \begin{pmatrix} \eta \sigma_i(y) u^i \\ 0 \end{pmatrix}$$

is homeomorphic for $y \in \mathbb{R}^d, \eta \in \mathbb{R}, u \in \{u \in \mathbb{R}^m : |u| \leq \delta\}$ and the Jacobian matrix

$$I + \begin{pmatrix} \eta \frac{\partial(\sigma_i(y))^1}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^1}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^1}{\partial y_d} u^i & (\sigma_i(y))^1 u^i \\ \eta \frac{\partial(\sigma_i(y))^2}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^2}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^2}{\partial y_d} u^i & (\sigma_i(y))^2 u^i \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \eta \frac{\partial(\sigma_i(y))^d}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^d}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^d}{\partial y_d} u^i & (\sigma_i(y))^d u^i \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is invertible for any y, η a.e. and $u \in \{u \in \mathbb{R}^m : |u| \leq \delta\}$. Then

- (i) the solution of Eq.(15) generates a global cocycle φ_t ,
- (ii) the solution of Eq.(16) generates a global cocycle ψ_t ,
- (iii) RDSs φ_t and ψ_t are conjugate with the cohomology H_0 .

Proof. By Theorem 3.1 in [14], the solution of Eq.(15) generates a global cocycle φ_t . However, by the assumptions of $a(x)$ and $\sigma_i(x)$, we only obtain that Eq.(16) generates a local cocycle denoted by ψ_t . To prove that ψ_t is global, we use conjugacy.

By the Itô-Ventzell formula in Appendix for $t \geq 0$, we have that

$$\begin{aligned}
dH_t(\omega, Y_t) &= \left(\frac{\partial}{\partial x} H_t \right) \left(\frac{\partial}{\partial x} H_t \right)^{-1} (\omega, Y_t) \left[a(H_t(\omega, Y_t)) - \Gamma_t(\omega, Y_t) \right] dt \\
&\quad + \Gamma_t(\omega, Y_t) dt + \sigma_i(H_t(\omega, Y_t)) b^i dt + \sigma_i(H_t(\omega, Y_t)) A^{ij} dW_t^j
\end{aligned}$$

$$\begin{aligned}
& + \int_{|u| \leq \delta} \sigma_i(H_t(\omega, Y_t)) u^i \tilde{N}_\kappa(dt, du) \\
& = a(H_t(\omega, Y_t))dt + \sigma_i(H_t(\omega, Y_t))d\hat{L}_t^i.
\end{aligned}$$

For $t < 0$, based on the similar deduction to the above one, the same result holds. Thus,

$$\varphi_t(\omega, \cdot) = H_0(\theta_t \omega, \cdot) \circ \psi_t(\omega, \cdot) \circ H_0(\omega, \cdot)^{-1},$$

and

$$\psi_t(\omega, \cdot) = H_0(\theta_t \omega, \cdot)^{-1} \circ \varphi_t(\omega, \cdot) \circ H_0(\omega, \cdot).$$

Since $\varphi_t(\omega, \cdot)$ is global, so is $\psi_t(\omega, \cdot)$. \square

3.2. Linearization of SDEs. In this subsection, we study the relation between a SDE and its linearized SDE. A stochastic version of the Hartman-Grobman theorem serves our purpose ([4, 11]). Using the promising technique of conjugacy for flows generated by Stratonovitch SDEs with Brownian motions and flows generated by RDEs, Imkeller and Lederer in [11] proved a Hartman-Grobman theorem for continuous cocycles. Here we prove a Hartman-Grobman theorem for discontinuous cocycles.

Theorem 3.4. *Assume that $a(x) \in \mathcal{C}_b^{2,\gamma}$, $\sigma_i(x) \in \mathcal{C}_b^{2,\gamma}$ for $i = 1, 2, \dots, m$, the mapping*

$$\begin{pmatrix} y \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} y \\ \eta \end{pmatrix} + \begin{pmatrix} \eta \sigma_i(y) u^i \\ 0 \end{pmatrix}$$

is homeomorphic for $y \in \mathbb{R}^d, \eta \in \mathbb{R}, u \in \{u \in \mathbb{R}^m : |u| \leq \delta\}$ and the Jacobian matrix

$$I + \begin{pmatrix} \eta \frac{\partial(\sigma_i(y))^1}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^1}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^1}{\partial y_d} u^i & (\sigma_i(y))^1 u^i \\ \eta \frac{\partial(\sigma_i(y))^2}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^2}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^2}{\partial y_d} u^i & (\sigma_i(y))^2 u^i \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \eta \frac{\partial(\sigma_i(y))^d}{\partial y_1} u^i & \eta \frac{\partial(\sigma_i(y))^d}{\partial y_2} u^i & \dots & \eta \frac{\partial(\sigma_i(y))^d}{\partial y_d} u^i & (\sigma_i(y))^d u^i \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

is invertible for any y, η a.e. and $u \in \{u \in \mathbb{R}^m : |u| \leq \delta\}$. Suppose $a(0) = 0$, $\sigma_i(0) = 0$ for $i = 1, 2, \dots, m$, and set $B_0 := \frac{\partial}{\partial x} a(0)$, $B_i := \frac{\partial}{\partial x} \sigma_i(0)$ for $i = 1, 2, \dots, m$.

$\varphi_t(\omega, \cdot)$ still denotes the RDS generated by Eq.(15). Let $\varphi_t^0(\omega, \cdot)$ be a RDS generated by

$$dx_t = B_0 x_t dt + B_i x_t d\hat{L}_t^i, \quad t \in \mathbb{R} \quad (17)$$

and suppose that all Lyapunov exponents of $\varphi_t^0(\omega, \cdot)$ are non-zero. Then there exists a measurable mapping $\varsigma : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfying the following properties

- (i) $x \mapsto \varsigma(\omega, x)$ is a homeomorphism of \mathbb{R}^d and $\varsigma(\omega, 0) = 0$,
- (ii) for $\omega \in \Omega$, it holds that

$$\varphi_t(\omega, x) = \varsigma(\theta_t \omega, \cdot) \circ \varphi_t^0(\omega, \cdot) \circ \varsigma(\omega, x)^{-1}, \quad t \in \mathbb{R}.$$

Proof. Step 1. Set

$$F(\omega, y) := \left(\frac{\partial}{\partial x} H_0 \right)^{-1} (\omega, y) \left[a(H_0(\omega, y)) - \Gamma_0(\omega, y) \right].$$

Then Eq.(16) can be rewritten as

$$\begin{cases} dY_t = F(\theta_t \omega, Y_t) dt, & t \in \mathbb{R}, \\ Y_0 = y. \end{cases} \quad (16)'$$

Moreover, $F(\cdot, 0) = 0$ and for $\omega \in \Omega$, $F(\omega, \cdot) \in \mathcal{C}^1(\mathbb{R}^d)$ by the assumptions for $a(x)$ and $\sigma_i(x)$, $i = 1, 2, \dots, m$.

Define

$$f(\omega) := \left(\frac{\partial}{\partial y} F \right) (\omega, 0).$$

Hence,

$$f(\omega) = \left(\frac{\partial}{\partial x} H_0 \right)^{-1} (\omega, 0) \left[B_0 \left(\frac{\partial}{\partial x} H_0 \right) (\omega, 0) - \left(\frac{\partial}{\partial x} \Gamma_0 \right) (\omega, 0) \right].$$

Introduce the following linear equation:

$$\begin{cases} dy_t = f(\theta_t \omega) y_t dt, & t \in \mathbb{R}, \\ y_0 = y. \end{cases} \quad (18)$$

By the similar result to Theorem 3.2 in [11] (Only the continuity of $\theta_t \omega$ in t is changed into right continuity with left limit), there exist measurable mappings $\varrho : \Omega \mapsto (0, \infty)$ and $\zeta : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ such that

- (i) $y \mapsto \zeta(\omega, y)$ is a homeomorphism of \mathbb{R}^d and $\zeta(\omega, 0) = 0$,
- (ii) for $\omega \in \Omega$, it holds that

$$\psi_t(\omega, y) = \zeta(\theta_t \omega, \cdot) \circ \psi_t^0(\omega, \cdot) \circ \zeta(\omega, y)^{-1}, \quad \tau_-^1(\omega, y) \leq t \leq \tau_+^1(\omega, y),$$

where $\psi_t(\omega, \cdot)$ and $\psi_t^0(\omega, \cdot)$ are the RDS generated by (16)' and (18), respectively, and

$$\begin{aligned} \tau_-^1(\omega, y) &:= \inf\{t < 0 : |\psi_s(\omega, y)| \leq \varrho(\theta_s \omega) \text{ for all } t \leq s \leq 0\}, \\ \tau_+^1(\omega, y) &:= \sup\{t > 0 : |\psi_s(\omega, y)| \leq \varrho(\theta_s \omega) \text{ for all } 0 \leq s \leq t\}. \end{aligned}$$

Next, for $t \geq \tau_+^1(\omega, y)$, take $\psi_{\tau_+^1}(\omega, y)$ and $\psi_{\tau_+^1}^0(\omega, y)$ as the initial values of (16) and (18), respectively, and consider (16) and (18) again. Using the cocycle property of solutions to (16) and (18), we obtain that

$$\begin{aligned} \psi_t(\omega, y) &= \psi_{t-\tau_+^1}(\theta_{\tau_+^1} \omega, \psi_{\tau_+^1}(\omega, y)) \\ &= \zeta(\theta_{t-\tau_+^1} \theta_{\tau_+^1} \omega, \cdot) \circ \psi_{t-\tau_+^1}^0(\theta_{\tau_+^1} \omega, \cdot) \circ \zeta(\theta_{\tau_+^1} \omega, \psi_{\tau_+^1}(\omega, y))^{-1} \\ &= \zeta(\theta_t \omega, \cdot) \circ \psi_{t-\tau_+^1}^0(\theta_{\tau_+^1} \omega, \cdot) \circ \zeta(\theta_{\tau_+^1} \omega, \cdot)^{-1} \circ \zeta(\theta_{\tau_+^1} \omega, \cdot) \\ &\quad \circ \psi_{\tau_+^1}^0(\omega, \cdot) \circ \zeta(\omega, y)^{-1} \\ &= \zeta(\theta_t \omega, \cdot) \circ \psi_{t-\tau_+^1}^0(\theta_{\tau_+^1} \omega, \cdot) \circ \psi_{\tau_+^1}^0(\omega, \cdot) \circ \zeta(\omega, y)^{-1} \\ &= \zeta(\theta_t \omega, \cdot) \circ \psi_t^0(\omega, \cdot) \circ \zeta(\omega, y)^{-1}, \end{aligned}$$

for $\tau_+^1(\omega, y) \leq t \leq \tau_+^2(\omega, y)$, where

$$\tau_+^2(\omega, y) = \sup\{t > \tau_+^1 : |\psi_s(\omega, y)| \leq \varrho(\theta_s \omega) \text{ for all } \tau_+^1 \leq s \leq t\}.$$

Doing this for $t \geq \tau_+^2(\omega, y)$ and $t \leq \tau_-^1(\omega, y)$, we see that for $\omega \in \Omega$,

$$\psi_t(\omega, y) = \zeta(\theta_t \omega, \cdot) \circ \psi_t^0(\omega, \cdot) \circ \zeta(\omega, y)^{-1}, \quad t \in \mathbb{R}.$$

Step 2. Consider the cocycles $\varphi_t^0(\omega, \cdot)$ and $\psi_t^0(\omega, \cdot)$ generated by Eq.(17) and Eq.(18), respectively. By the Itô-Ventzell formula in Appendix for $t \geq 0$, we get that

$$d \left(\frac{\partial}{\partial x} H_t \right) (\cdot, 0) y_t = \left(\frac{\partial}{\partial x} H_t \right) (\cdot, 0) \left(\frac{\partial}{\partial x} H_t \right)^{-1} (\cdot, 0) \left[B_0 \left(\frac{\partial}{\partial x} H_t \right) (\cdot, 0) \right]$$

$$\begin{aligned}
& - \left(\frac{\partial}{\partial x} \Gamma_t \right) (\cdot, 0) \Big] y_t dt + \left(\frac{\partial}{\partial x} \Gamma_t \right) (\cdot, 0) y_t dt \\
& + B_i \left(\frac{\partial}{\partial x} H_t \right) (\cdot, 0) y_t d\hat{L}_t^i \\
& = B_0 \left(\frac{\partial}{\partial x} H_t \right) (\cdot, 0) y_t dt + B_i \left(\frac{\partial}{\partial x} H_t \right) (\cdot, 0) y_t d\hat{L}_t^i.
\end{aligned}$$

For $t < 0$, in terms of the similar deduction to the above one, the result also holds.

Thus $\varphi_t^0(\omega, \cdot)$ and $\psi_t^0(\omega, \cdot)$ are conjugate. $\left(\frac{\partial}{\partial x} H_0 \right)^{-1}(\cdot, 0)$ is a cohomology of $\psi_t^0(\omega, \cdot)$ and $\varphi_t^0(\omega, \cdot)$.

Step 3. Combining Proposition 3.3 with **Steps 1-2**, we obtain that

$$\begin{aligned}
\varphi_t(\omega, \cdot) &= H_0(\theta_t \omega, \cdot) \circ \psi_t(\omega, \cdot) H_0(\omega, \cdot)^{-1} \\
&= H_0(\theta_t \omega, \cdot) \circ \zeta(\theta_t \omega, \cdot) \circ \psi_t^0(\omega, \cdot) \circ \zeta(\omega, \cdot)^{-1} \circ H_0(\omega, \cdot)^{-1} \\
&= H_0(\theta_t \omega, \cdot) \circ \zeta(\theta_t \omega, \cdot) \circ \left(\frac{\partial}{\partial x} H_0 \right)^{-1}(\theta_t \omega, 0) \circ \varphi_t^0(\omega, \cdot) \\
&\quad \circ \left(\frac{\partial}{\partial x} H_0 \right)(\omega, 0) \circ \zeta(\omega, \cdot)^{-1} \circ H_0(\omega, \cdot)^{-1}.
\end{aligned}$$

Set

$$\varsigma(\omega, \cdot) := H_0(\omega, \cdot) \circ \zeta(\omega, \cdot) \circ \left(\frac{\partial}{\partial x} H_0 \right)^{-1}(\omega, 0) \cdot.$$

We thus have

$$\varphi_t(\omega, x) = \varsigma(\theta_t \omega, \cdot) \circ \varphi_t^0(\omega, \cdot) \circ \varsigma(\omega, x)^{-1}.$$

The proof is completed. \square

Remark 3.5. Since ψ_t and ψ_t^0 are locally conjugate via ζ in **Step 1**, the result in the above theorem is local.

We present an example to demonstrate the above theorem.

Example 3.6. Take $a(x) = \beta x - x^l$ and $\sigma(x) = x$, where $\beta = \alpha + \sigma^2/2$. Here α is a real parameter and σ is a positive parameter. Let $l > 1$ be an integer. Consider the following scalar nonlinear stochastic equation

$$dX_t = (\beta X_t - X_t^l)dt + X_t d\hat{L}_t, \quad t \in \mathbb{R}. \quad (19)$$

Lyapunov stability for this equation has been studied in a special case where the Lévy process is replaced by a compound Poisson process in [8].

In this example, $B_0 = \frac{\partial}{\partial x} a(0) = \beta$, and $B = \frac{\partial}{\partial x} \sigma(0) = 1$. Thus, by Theorem 3.4, the RDS $\varphi_t(\omega, \cdot)$ generated by Eq.(19) is conjugate to the RDS $\varphi_t^0(\omega, \cdot)$ generated by the linear stochastic equation

$$dx_t = \beta x_t dt + x_t d\hat{L}_t, \quad t \in \mathbb{R}. \quad (20)$$

4. CONJUGACY FOR MARCUS SDEs AND RDEs, AND EXISTENCE OF GLOBAL ATTRACTORS

In this section, we first introduce Marcus SDEs, then we prove the conjugacy for a Marcus SDE and a related RDE and the existence of the global attractor for the Marcus SDE.

4.1. Conjugacy for Marcus SDEs and RDEs. A Marcus canonical SDE was introduced in [16]. For the Lévy process \hat{L}_t in Section 3.1, this SDE is as follows:

$$d\bar{X}_t = \bar{a}(\bar{X}_t)dt + \bar{\sigma}_i(\bar{X}_t) \diamond d\hat{L}_t^i, \quad t \in \mathbb{R}, \quad (21)$$

where $\bar{a}, \bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_m$ are vector fields on \mathbb{R}^d , and “ \diamond ” denotes the Marcus differential ([1, 14]). The precise definition is as follows:

$$d\bar{X}_t = \begin{cases} \bar{a}(\bar{X}_t)dt + \bar{\sigma}_i(\bar{X}_t) \circ d\hat{L}_c^i(t) + \bar{\sigma}_i(\bar{X}_{t-})d\hat{L}_d^i(t) \\ + \left\{ \Phi(\Delta\hat{L}_t, \bar{X}_{t-}) - \bar{X}_{t-} - \bar{\sigma}_i(\bar{X}_{t-})\Delta\hat{L}_t^i \right\}, & t \geq 0, \\ \bar{a}(\bar{X}_t)dt + \bar{\sigma}_i(\bar{X}_t) \circ d\hat{L}_c^i(t) + \bar{\sigma}_i(\bar{X}_{t-})d\hat{L}_d^i(t) \\ + \left\{ \Phi(\Delta\hat{L}_t, \bar{X}_{t+}) - \bar{X}_{t+} - \bar{\sigma}_i(\bar{X}_{t+})\Delta\hat{L}_t^i \right\}, & t \leq 0, \end{cases}$$

where “ \circ ” stands for the Stratonovitch differential, $\hat{L}_c(t)$ and $\hat{L}_d(t)$ are continuous and purely discontinuous parts of the Lévy process $\hat{L}(t)$, respectively,

$$\Delta\hat{L}_t := \begin{cases} \hat{L}_t - \hat{L}_{t-}, & t \geq 0, \\ \hat{L}_t - \hat{L}_{t+}, & t \leq 0, \end{cases}$$

and $\Phi(z, x)$ is the solution flow of the following ordinary differential equation

$$\begin{cases} \frac{\partial \Phi(z, x)}{\partial z_i} = \bar{\sigma}_i(\Phi(z, x)), & i = 1, 2, \dots, m, \\ \Phi(0, x) = x. \end{cases} \quad (22)$$

Remark 4.1. If the Lévy process \hat{L}_t has no discontinuous part, Eq.(21) changes into a Stratonovitch SDE, and $\Phi(z, x)$ is the same as the auxiliary function in the Doss-Sussmann flow decomposition ([11, 12]).

Assume that $\bar{\sigma}_i(x)$ is smooth for $i = 1, 2, \dots, m$ and the Lie bracket $[\bar{\sigma}_i, \bar{\sigma}_j] = 0$ in the sense of differential geometry for $i, j = 1, 2, \dots, m, i \neq j$. Then Eq.(22) possesses a solution denoted by $\Phi(z, x)$ and $x \mapsto \Phi(z, x)$ is a diffeomorphism of \mathbb{R}^d , as in [22].

For $\mu > 0$, introduce the following process

$$Z_t = e^{-\mu t} \int_{-\infty}^t e^{\mu s} d\hat{L}_s, \quad t \in \mathbb{R}.$$

Lemma 4.2. Z_t has the following properties:

(i)

$$dZ_t = -\mu Z_t dt + d\hat{L}_t, \quad t \in \mathbb{R},$$

(ii)

$$Z_{s+t}(\omega) = Z_t(\theta_s \omega), \quad s, t \in \mathbb{R}.$$

Proof. Set $\bar{\bar{L}}_t := \int_{-\infty}^t e^{\mu s} d\hat{L}_s$, and then $Z_t = e^{-\mu t} \bar{\bar{L}}_t$. By the Itô-Ventzell formula in Appendix for $t \geq 0$, one obtains that

$$\begin{aligned} dZ_t &= e^{-\mu t} e^{\mu t} b dt + e^{-\mu t} e^{\mu t} \text{Ad}W_t - \mu e^{-\mu t} \bar{\bar{L}}_t dt \\ &\quad + \int_{|u| \leq \delta} \left(e^{-\mu t} (\bar{\bar{L}}_t + e^{\mu t} u) - e^{-\mu t} \bar{\bar{L}}_t \right) \tilde{N}_\kappa(dt, du) \\ &\quad + \int_{|u| \leq \delta} \left(e^{-\mu t} (\bar{\bar{L}}_t + e^{\mu t} u) - e^{-\mu t} \bar{\bar{L}}_t - e^{-\mu t} e^{\mu t} u \right) \nu(du) dt \\ &= -\mu Z_t dt + d\hat{L}_t. \end{aligned}$$

For $t < 0$, by the similar deduction to the above one, we could prove the same result. This yields (i).

To prove (ii), it holds that

$$\begin{aligned} Z_{s+t}(\omega) &= e^{-\mu(s+t)} \int_{-\infty}^{s+t} e^{\mu r} d\hat{L}_r \\ &= e^{-\mu(s+t)} \int_{-\infty}^t e^{\mu(s+u)} d(\hat{L}_{s+u} - \hat{L}_s) \\ &= e^{-\mu t} \int_{-\infty}^t e^{\mu u} d\hat{L}_u(\theta_s \omega) = Z_t(\theta_s \omega). \end{aligned}$$

The proof is completed. \square

Denote

$$\bar{H}_t(x) := \Phi(Z_t, x), \quad t \in \mathbb{R}.$$

By the above Lemma,

$$\bar{H}_{s+t}(\omega, x) = \bar{H}_t(\theta_s \omega, x)$$

and by Theorem 4.4.7 or Theorem 4.4.28 in [1],

$$\begin{aligned} d\bar{H}_t &= \bar{\sigma}_i(\bar{H}_t) \diamond dZ_t^i \\ &= -\mu \bar{\sigma}_i(\bar{H}_t) Z_t^i dt + \bar{\sigma}_i(\bar{H}_t) \diamond d\hat{L}_t^i. \end{aligned}$$

Introduce the following RDE

$$d\bar{Y}_t = \left(\frac{\partial}{\partial x} \bar{H}_t \right)^{-1} (\omega, \bar{Y}_t) \left[\bar{a}(\bar{H}_t(\omega, \bar{Y}_t)) + \mu \bar{\sigma}_i(\bar{H}_t(\omega, \bar{Y}_t)) Z_t^i \right] dt, \quad t \in \mathbb{R}. \quad (23)$$

Proposition 4.3. *RDSs $\bar{\varphi}_t(\omega, \cdot)$ and $\bar{\psi}_t(\omega, \cdot)$ are conjugate with cohomology \bar{H}_0 , where $\bar{\varphi}_t(\omega, \cdot), \bar{\psi}_t(\omega, \cdot)$ are RDSs generated by Eq.(21) and Eq.(23), respectively.*

Proof. We firstly show that the solutions of Eq.(21) and Eq.(23) generate global RDSs. By Theorem 3.16 in [14], the solution of Eq.(21) generates a global cocycle $\bar{\varphi}_t(\omega, \cdot)$. To prove that $\bar{\psi}_t(\omega, \cdot)$ is global, we use conjugacy.

Rewrite $d\bar{H}_t$ as the Itô form for $t \geq 0$

$$\begin{aligned} d\bar{H}_t &= -\mu \bar{\sigma}_i(\bar{H}_t) Z_t^i dt + \bar{\sigma}_i(\bar{H}_t) b^i dt + \bar{\sigma}_i(\bar{H}_t) A^{ij} dW_t^j \\ &\quad + \frac{1}{2} \left(\frac{\partial \bar{\sigma}_i}{\partial x_k} \right) (\bar{H}_t) (\bar{\sigma}_j(\bar{H}_t))^k A^{il} A^{jl} dt \end{aligned}$$

$$\begin{aligned}
& + \int_{|u| \leq \delta} (\Phi(u, \bar{H}_{t-}) - \bar{H}_{t-}) \tilde{N}_\kappa(dt, du) \\
& + \int_{|u| \leq \delta} (\Phi(u, \bar{H}_{t-}) - \bar{H}_{t-} - \bar{\sigma}_i(\bar{H}_{t-})u^i) \nu(du)dt,
\end{aligned}$$

and for $t \leq 0$

$$\begin{aligned}
d\bar{H}_t &= -\mu \bar{\sigma}_i(\bar{H}_t) Z_t^i dt - \bar{\sigma}_i(\bar{H}_t) (b^-)^i dt - \bar{\sigma}_i(\bar{H}_t) (A^-)^{ij} d(W^-)^j_t \\
& + \frac{1}{2} \left(\frac{\partial \bar{\sigma}_i}{\partial x_k} \right) (\bar{H}_t) (\bar{\sigma}_j(\bar{H}_t))^k (A^-)^{il} (A^-)^{jl} dt \\
& + \int_{|u| \leq \delta} (\Phi(u, \bar{H}_{t+}) - \bar{H}_{t+}) \tilde{N}_{\kappa^-}(dt, du) \\
& + \int_{|u| \leq \delta} (\Phi(u, \bar{H}_{t+}) - \bar{H}_{t+} - \bar{\sigma}_i(\bar{H}_{t+})u^i) \nu^-(du)dt.
\end{aligned}$$

Thus, the Itô-Ventzell formula in Appendix for $t \geq 0$ implies that

$$\begin{aligned}
d\bar{H}_t(\omega, \bar{Y}_t) &= \left(\frac{\partial}{\partial x} \bar{H}_t \right) \left(\frac{\partial}{\partial x} \bar{H}_t \right)^{-1} (\omega, \bar{Y}_t) \left[\bar{a}(\bar{H}_t(\omega, \bar{Y}_t)) + \mu \bar{\sigma}_i(\bar{H}_t(\omega, \bar{Y}_t)) Z_t^i \right] dt \\
& - \mu \bar{\sigma}_i(\bar{H}_t(\omega, \bar{Y}_t)) Z_t^i dt + \bar{\sigma}_i(\bar{H}_t(\omega, \bar{Y}_t)) b^i dt + \bar{\sigma}_i(\bar{H}_t(\omega, \bar{Y}_t)) A^{ij} dW_t^j \\
& + \frac{1}{2} \left(\frac{\partial \bar{\sigma}_i}{\partial x_k} \right) (\bar{H}_t(\omega, \bar{Y}_t)) (\bar{\sigma}_j(\bar{H}_t(\omega, \bar{Y}_t)))^k A^{il} A^{jl} dt \\
& + \int_{|u| \leq \delta} (\Phi(u, \bar{H}_{t-}(\omega, \bar{Y}_t)) - \bar{H}_{t-}(\omega, \bar{Y}_t)) \tilde{N}_\kappa(dt, du) \\
& + \int_{|u| \leq \delta} (\Phi(u, \bar{H}_{t-}(\omega, \bar{Y}_t)) - \bar{H}_{t-}(\omega, \bar{Y}_t) - \bar{\sigma}_i(\bar{H}_{t-}(\omega, \bar{Y}_t))u^i) \nu(du)dt \\
& = \bar{a}(\bar{H}_t(\omega, \bar{Y}_t))dt + \bar{\sigma}_i(\bar{H}_t(\omega, \bar{Y}_t)) \diamond d\hat{L}_t^i.
\end{aligned}$$

For $t < 0$, by the similar deduction to the above one, the same result holds. Thus,

$$\bar{\varphi}_t(\omega, \cdot) = \bar{H}_0(\theta_t \omega, \cdot) \circ \bar{\psi}_t(\omega, \cdot) \circ \bar{H}_0(\omega, \cdot)^{-1},$$

and

$$\bar{\psi}_t(\omega, \cdot) = \bar{H}_0(\theta_t \omega, \cdot)^{-1} \circ \bar{\varphi}_t(\omega, \cdot) \circ \bar{H}_0(\omega, \cdot).$$

Since $\bar{\varphi}_t(\omega, \cdot)$ is global, so is $\bar{\psi}_t(\omega, \cdot)$. □

We present two examples to explain what the cohomology \bar{H}_0 is.

Example 4.4. Take $\bar{\sigma}_i(x) = \bar{\sigma}_i x$ for $i = 1, 2, \dots, m$, where $\bar{\sigma}_i \in \mathbb{R}^{d \times d}$ and $\bar{\sigma}_i \bar{\sigma}_j = \bar{\sigma}_j \bar{\sigma}_i$, $i \neq j$. Then Eq.(22) changes into

$$\begin{cases} \frac{\partial \Phi(z, x)}{\partial z_i} = \bar{\sigma}_i \Phi(z, x), & i = 1, 2, \dots, m, \\ \Phi(0, x) = x. \end{cases}$$

Solving it, we obtain that $\Phi(z, x) = x \exp\{\bar{\sigma}_i z^i\}$. So,

$$\begin{aligned}
\bar{H}_t(\cdot, x) &= \Phi(Z_t, x) = x \exp\{\bar{\sigma}_i Z_t^i\} = x \exp\{\bar{\sigma}_i Z_0^i\} \circ \theta_t \cdot \\
&= \bar{H}_0(\theta_t \cdot, x).
\end{aligned}$$

In this case, the RDE (23) becomes

$$d\bar{Y}_t = \exp\{-\bar{\sigma}_i Z_t^i\} \left[\bar{a}(\bar{Y}_t \exp\{\bar{\sigma}_i Z_t^i\}) + \mu \bar{\sigma}_i \bar{Y}_t \exp\{\bar{\sigma}_i Z_t^i\} Z_t^i \right] dt, \quad t \in \mathbb{R}.$$

Example 4.5. Take $\bar{\sigma}_i(x) = \bar{\sigma}_i x + \beta_i$ for $i = 1, 2, \dots, m$, where $\beta_i \in \mathbb{R}^d$, $\bar{\sigma}_i \in \mathbb{R}^{d \times d}$ and $\bar{\sigma}_i \bar{\sigma}_j = \bar{\sigma}_j \bar{\sigma}_i$, $\bar{\sigma}_i \beta_j = \bar{\sigma}_j \beta_i$, $i \neq j$. Thus the solution of Eq.(22) is

$$\Phi(z, x) = x \exp\{\bar{\sigma}_i z^i\} + \bar{\sigma}_i^{-1} (\exp\{\bar{\sigma}_i z^i\} - I) \beta_i,$$

where $\bar{\sigma}_i^{-1}$ is the pseudo-inverse of the matrix $\bar{\sigma}_i$.

Hence

$$\begin{aligned} \bar{H}_t(\cdot, x) &= \Phi(Z_t, x) = x \exp\{\bar{\sigma}_i Z_t^i\} + \bar{\sigma}_i^{-1} (\exp\{\bar{\sigma}_i Z_t^i\} - I) \beta_i \\ &= [x \exp\{\bar{\sigma}_i Z_0^i\} + \bar{\sigma}_i^{-1} (\exp\{\bar{\sigma}_i Z_0^i\} - I) \beta_i] \circ \theta_t \cdot \\ &= \bar{H}_0(\theta_t \cdot, x). \end{aligned}$$

Finally, the RDE (23) is written as

$$d\bar{Y}_t = \exp\{-\bar{\sigma}_i Z_t^i\} \left[\bar{a}(\bar{H}_t(\omega, \bar{Y}_t)) + \mu \bar{\sigma}_i \bar{H}_t(\omega, \bar{Y}_t) Z_t^i + \mu \beta_i Z_t^i \right] dt, \quad t \in \mathbb{R}.$$

4.2. Existence of global attractors. We need some concepts. The function $V : \mathbb{R}^d \mapsto \mathbb{R}_+$ is called the Lyapunov function of

$$d\bar{y}_t = \bar{a}(\bar{y}_t) dt, \quad (24)$$

if there exists $\alpha > 0$ such that

$$\limsup_{|y| \rightarrow \infty} \langle \nabla \log V(y), \bar{a}(y) \rangle \leq -\alpha. \quad (25)$$

A function $U : \mathbb{R}^d \mapsto \mathbb{R}_+$ is said to *preserve temperedness* if $U^{-1}(G)$ is tempered for tempered G . Finally, a function $k : \mathbb{R}^m \mapsto \mathbb{R}$ is said to be *subexponentially growing* if there exists $c > 0$ such that $e^{-c|z|} k(z)$ is bounded.

Proposition 4.6. Let V be a Lyapunov function of Eq.(24) and assume that V preserves temperedness. Set

$$l(z, y) := \left(\frac{\partial \Phi}{\partial y} \right)^{-1} (z, y) \bar{\sigma}_i (\Phi(z, y)) z^i, \quad z \in \mathbb{R}^m, y \in \mathbb{R}^d.$$

Assume that there are subexponentially growing functions k_1 and k_2 such that

$$\limsup_{|y| \rightarrow \infty} \sup_{z \in \mathbb{R}^m} \left| \left\langle \nabla \log V(y), \frac{l(z, y)}{k_1(z)} \right\rangle \right| = 0, \quad (26)$$

$$\limsup_{|y| \rightarrow \infty} \sup_{z \in \mathbb{R}^m} \frac{\left\langle \nabla \log V(y), \bar{a}(y) - \left(\frac{\partial \Phi}{\partial y} \right)^{-1} (z, y) \bar{a}(\Phi(z, y)) \right\rangle}{|\langle \nabla \log V(y), \bar{a}(y) \rangle| k_2(z)} \leq 1, \quad (27)$$

and

$$\lim_{z \rightarrow 0} k_2(z) = 0.$$

Then $\bar{\psi}_t$ has a global random attractor.

The proof is similar to one for Corollary 2.1 in [12] and we thus omit it.

Now, we state and prove the main result in this subsection.

Theorem 4.7. *Under the conditions of Proposition 4.6, $\bar{\varphi}_t$ has a global random attractor.*

Proof. By the above Proposition, we know that $\bar{\psi}_t$ has a global random attractor $A(\omega)$. Then by definition

$$\bar{\psi}_t(\omega, A(\omega)) = A(\theta_t \omega).$$

Define $B(\omega) := \bar{H}_0(\omega, A(\omega))$. Therefore

$$\begin{aligned} \bar{\varphi}_t(\omega, B(\omega)) &= \bar{\varphi}_t(\omega, \cdot) \circ \bar{H}_0(\omega, A(\omega)) \\ &= \bar{\varphi}_t(\omega, \cdot) \circ \bar{H}_0(\omega, \cdot) \circ A(\omega) \\ &= \bar{H}_0(\theta_t \omega, \cdot) \circ \bar{\psi}_t(\omega, \cdot) \circ \bar{H}_0(\omega, \cdot)^{-1} \circ \bar{H}_0(\omega, \cdot) \circ A(\omega) \\ &= \bar{H}_0(\theta_t \omega, \cdot) \circ \bar{\psi}_t(\omega, \cdot) \circ A(\omega) \\ &= \bar{H}_0(\theta_t \omega, A(\theta_t \omega)) \\ &= B(\theta_t \omega). \end{aligned}$$

For any random tempered bounded set $G(\omega)$, by the assumption on $\bar{\sigma}_i$ for $i = 1, 2, \dots, m$, $\bar{H}_0(\omega, \cdot)^{-1} \circ G(\omega)$ is tempered. Thus, by Definition 2.4,

$$\lim_{t \rightarrow +\infty} \text{dist}(\bar{\psi}_t(\theta_{-t} \omega, \cdot) \circ \bar{H}_0(\theta_{-t} \omega, \cdot)^{-1} \circ G(\theta_{-t} \omega), A(\omega)) = 0.$$

Since, by conjugacy and the assumption on $\bar{\sigma}_i$ for $i = 1, 2, \dots, m$,

$$\begin{aligned} &\text{dist}(\bar{\varphi}_t(\theta_{-t} \omega, G(\theta_{-t} \omega)), B(\omega)) \\ &= \text{dist}(\bar{\varphi}_t(\theta_{-t} \omega, \cdot) \circ G(\theta_{-t} \omega), \bar{H}_0(\omega, A(\omega))) \\ &= \text{dist}(\bar{H}_0(\omega, \cdot) \circ \bar{\psi}_t(\theta_{-t} \omega, \cdot) \circ \bar{H}_0(\theta_{-t} \omega, \cdot)^{-1} \circ G(\theta_{-t} \omega), \bar{H}_0(\omega, A(\omega))) \\ &\leq \text{dist}(\bar{\psi}_t(\theta_{-t} \omega, \cdot) \circ \bar{H}_0(\theta_{-t} \omega, \cdot)^{-1} \circ G(\theta_{-t} \omega), A(\omega)), \end{aligned}$$

we obtain that

$$\lim_{t \rightarrow +\infty} \text{dist}(\bar{\varphi}_t(\theta_{-t} \omega, G(\theta_{-t} \omega)), B(\omega)) = 0,$$

for all tempered random bounded set $G(\omega) \subset \mathbb{R}^d$.

Finally, by Definition 2.4, $B(\omega)$ is a global random attractor for $\bar{\varphi}_t$. This completes the proof. \square

In the following example, we apply the above theorem to show the existence of a global attractor for the stochastic Duffing-van der Pol equations with Lévy processes.

Example 4.8. *Consider the following Duffing-van der Pol system with Lévy processes:*

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = [\gamma_1 x_1(t) + \gamma_2 x_2(t) - x_1(t)^3 - x_1(t)^2 x_2(t)]dt \\ \quad + \sigma_1 x_1(t) \diamond d\hat{L}_t^1 + \sigma_2 \diamond d\hat{L}_t^2, \end{cases}$$

where $\gamma_1, \gamma_2, \sigma_1, \sigma_2 \in \mathbb{R}^1$ and $\hat{L}_t = (\hat{L}_t^1, \hat{L}_t^2)$ is a 2-dimensional Lévy process as defined in Section 3.1. This system has been studied in special cases: the Lévy process is replaced by the Brownian motion in [12, 21] and by the Poisson process in [25]. Set

$$\begin{aligned} \bar{X}_1(t) &:= x_1(t), \\ \bar{X}_2(t) &:= x_2(t) - \gamma_2 x_1(t) + \frac{1}{3} x_1(t)^3. \end{aligned}$$

Then the above system is transformed into

$$d\bar{X}_t = \bar{a}(\bar{X}_t)dt + (\bar{\sigma}_i \bar{X}_t + \beta_i) \diamond d\hat{L}_t^i, \quad t \in \mathbb{R},$$

where

$$\bar{a}(y) = \begin{pmatrix} \gamma_2 y_1 - \frac{1}{3} y_1^3 + y_2 \\ \gamma_1 y_1 - y_1^3 \end{pmatrix},$$

$$\bar{\sigma}_1 = \begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix}, \quad \beta_1 = 0, \quad \bar{\sigma}_2 = 0, \quad \text{and } \beta_2 = \begin{pmatrix} 0 \\ \sigma_2 \end{pmatrix}.$$

For the deterministic equation

$$d\bar{y}_t = \bar{a}(\bar{y}_t)dt,$$

the function

$$V(y) = \frac{7}{24} y_1^4 + \frac{1}{4} y_1^2 + \frac{1}{4} y_2^2 + \frac{1}{2} (y_1 - y_2)^2$$

is a Lyapunov function. Indeed,

$$\nabla V(y) = \begin{pmatrix} \frac{7}{6} y_1^3 + \frac{3}{2} y_1 - y_2 \\ \frac{3}{2} y_2 - y_1 \end{pmatrix},$$

and

$$\begin{aligned} \langle \nabla V(y), \bar{a}(y) \rangle &= -\frac{7}{18} y_1^6 + \left(\frac{7}{6} \gamma_2 + \frac{1}{2} \right) y_1^4 + \left(\frac{3}{2} \gamma_2 - \gamma_1 \right) y_1^2 \\ &\quad + \left(\frac{3}{2} - \gamma_2 + \frac{3}{2} \gamma_1 \right) y_1 y_2 - y_2^2. \end{aligned}$$

Thus, there exists $\eta > 0$ such that

$$\limsup_{|y| \rightarrow \infty} \frac{\langle \nabla V(y), \bar{a}(y) \rangle}{\kappa(y)} \leq -\eta, \quad (28)$$

where $\kappa(y) = y_1^6 + y_2^2$. Thus, $V(y)$ is a Lyapunov function.

Noticing that

$$\exp\{\bar{\sigma}_1 z^1\} = \begin{pmatrix} 1 & 0 \\ \sigma_1 z^1 & 1 \end{pmatrix}, \quad \exp\{-\bar{\sigma}_1 z^1\} = \begin{pmatrix} 1 & 0 \\ -\sigma_1 z^1 & 1 \end{pmatrix},$$

by Example 4.5, we get

$$\begin{aligned} \Phi(z, y) &= y \exp\{\bar{\sigma}_1 z^1\} + \beta_2 z^2 \\ &= \begin{pmatrix} y_1 \\ \sigma_1 y_1 z^1 + y_2 + \sigma_2 z^2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} l(z, y) &= \left(\frac{\partial \Phi}{\partial y} \right)^{-1} (z, y) [\bar{\sigma}_1 (\Phi(z, y)) z^1 + \bar{\sigma}_2 (\Phi(z, y)) z^2] \\ &= \exp\{-\bar{\sigma}_1 z^1\} [\bar{\sigma}_1 (y \exp\{\bar{\sigma}_1 z^1\} + \beta_2 z^2) z^1 + \beta_2 z^2] \\ &= \bar{\sigma}_1 y z^1 + \exp\{-\bar{\sigma}_1 z^1\} \beta_2 z^2 \\ &= \begin{pmatrix} 0 \\ \sigma_1 y_1 z^1 + \sigma_2 z^2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned}\langle \nabla V(y), l(z, y) \rangle &= \left(\frac{3}{2} y_2 - y_1 \right) (\sigma_1 y_1 z^1 + \sigma_2 z^2) \\ &= \sigma_1 \left(\frac{3}{2} y_1 y_2 - y_1^2 \right) z^1 + \sigma_2 \left(\frac{3}{2} y_2 - y_1 \right) z^2.\end{aligned}$$

Set $k_1(z) := C|z|$. Then $k_1(z)$ is subexponentially growing and

$$\limsup_{|y| \rightarrow \infty} \sup_{z \in \mathbb{R}^m} \left| \left\langle \nabla \log V(y), \frac{l(z, y)}{k_1(z)} \right\rangle \right| = 0.$$

Thus, the condition (26) in Proposition 4.6 is satisfied.

To justify (27), we decompose $\bar{a}(y)$ into two parts:

$$\bar{a}(y) = Ay + \begin{pmatrix} -\frac{1}{3}y_1^3 \\ -y_1^3 \end{pmatrix},$$

where $A := \begin{pmatrix} \gamma_2 & 1 \\ \gamma_1 & 0 \end{pmatrix}$. Hence,

$$\begin{aligned}\bar{a}(y) - \left(\frac{\partial \Phi}{\partial y} \right)^{-1}(z, y) \bar{a}(\Phi(z, y)) &= Ay - \left(\frac{\partial \Phi}{\partial y} \right)^{-1}(z, y) A \Phi(z, y) \\ &\quad + \begin{pmatrix} -\frac{1}{3}y_1^3 \\ -y_1^3 \end{pmatrix} - \left(\frac{\partial \Phi}{\partial y} \right)^{-1}(z, y) \begin{pmatrix} -\frac{1}{3}y_1^3 \\ -y_1^3 \end{pmatrix} \\ &= Ay - \exp\{-\bar{\sigma}_1 z^1\} A \exp\{\bar{\sigma}_1 z^1\} y - \exp\{-\bar{\sigma}_1 z^1\} A \beta_2 z^2 \\ &\quad + (I - \exp\{-\bar{\sigma}_1 z^1\}) \begin{pmatrix} -\frac{1}{3}y_1^3 \\ -y_1^3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ \gamma_2 + \sigma_1 z^1 & 1 \end{pmatrix} y \sigma_1 z^1 - \begin{pmatrix} 1 \\ -\sigma_1 z^1 \end{pmatrix} \sigma_2 z^2 \\ &\quad + \begin{pmatrix} 0 \\ -\frac{1}{3}y_1^3 \end{pmatrix} \sigma_1 z^1 \\ &=: S_1 + S_2 + S_3.\end{aligned}$$

Define

$$\begin{aligned}k_2(z) : &= (\sqrt{2} + |\gamma_2| + |\sigma_1||z|)|\sigma_1||z| \\ &\quad + (1 + |\sigma_1||z|)|\sigma_2||z| + |\sigma_1||z|.\end{aligned}$$

Then $k_2(z)$ is subexponentially growing and

$$\lim_{z \rightarrow 0} k_2(z) = 0.$$

Moreover,

$$\begin{aligned}\frac{\left\langle \nabla \log V(y), \bar{a}(y) - \left(\frac{\partial \Phi}{\partial y} \right)^{-1}(z, y) \bar{a}(\Phi(z, y)) \right\rangle}{|\langle \nabla \log V(y), \bar{a}(y) \rangle| k_2(z)} &= \frac{\langle \nabla \log V(y), S_1 + S_2 + S_3 \rangle}{|\langle \nabla \log V(y), \bar{a}(y) \rangle| k_2(z)} \\ &\leq \frac{|\nabla V(y)||y|}{|\langle \nabla V(y), \bar{a}(y) \rangle|} + \frac{|\nabla V(y)|}{|\langle \nabla V(y), \bar{a}(y) \rangle|}\end{aligned}$$

$$+ \frac{|(\frac{3}{2}y_2 - y_1)\frac{1}{3}y_1^3|}{|\langle \nabla V(y), \bar{a}(y) \rangle|}.$$

It follows from (28) that there exists a constant $C > 0$ such that

$$|\langle \nabla V(y), \bar{a}(y) \rangle| \leq C \kappa(y).$$

Thus, by simple calculation, we obtain that

$$\limsup_{|y| \rightarrow \infty} \sup_{z \in \mathbb{R}^m} \frac{\left\langle \nabla \log V(y), \bar{a}(y) - \left(\frac{\partial \Phi}{\partial y}\right)^{-1}(z, y) \bar{a}(\Phi(z, y)) \right\rangle}{|\langle \nabla \log V(y), \bar{a}(y) \rangle| k_2(z)} \leq 1.$$

By Theorem 4.7, the stochastic Duffing-van der Pol equation has a global attractor.

5. APPENDIX

In this section, we recall the Itô-Ventzell formula in the context of SDEs with Lévy processes (see [18]), and then provide a proof for the formula (14) in Section 3.

The Itô-Ventzell formula:

Let $(\mathbb{U}, \mathcal{U})$ be a measurable space and n be a σ -finite measure on it. Let \mathbb{U}_0 be a set in \mathcal{U} such that $n(\mathbb{U} - \mathbb{U}_0) < \infty$.

Consider two processes with jumps

$$\begin{cases} \eta_t = \eta_0 + \int_0^t e_s \, ds + \int_0^t f_s \, dW_s + \int_0^{t+} \int_{\mathbb{U}_0} g(s-, u) \tilde{N}_q(ds, du), \\ \xi_t(x) = \xi_0(x) + \int_0^t E_s(x) \, ds + \int_0^t F_s^l(x) \, dW_s^l + \int_0^{t+} \int_{\mathbb{U}_0} G(s-, x, u) \tilde{N}_q(ds, du), \end{cases}$$

where $\eta, e, f, g(\cdot, u)$ for $u \in \mathbb{U}_0$ are predictable processes valued in $\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^{d \times m}, \mathbb{R}^d$, respectively, and $\xi(x), E_s(x), F_s^l(x), G(\cdot, x, u)$ for $x \in \mathbb{R}^d, u \in \mathbb{U}_0$ and $l = 1, 2, \dots, d$ are real predictable processes; $\{q_t, t \geq 0\}$ is a stationary \mathcal{F}_0^t -adapted Poisson point process with values in \mathbb{U} and characteristic measure n . Let $N_q((0, t], du)$ be the counting measure of q_t such that $\mathbb{E}N_q((0, t], A) = tn(A)$ for $A \in \mathcal{U}$. Denote

$$\tilde{N}_q((0, t], du) := N_q((0, t], du) - tn(du),$$

the compensator of q_t . Moreover e, f, g and $E(x), F^l(x), G(\cdot, x, \cdot)$ satisfy the following integrable conditions: for $t \in \mathbb{R}_+$

$$\begin{aligned} \int_0^t |e_s| \, ds &< \infty, \int_0^t |f_s|^2 \, ds < \infty, \int_0^{t+} \int_{\mathbb{U}_0} |g(s-, u)|^2 n(du) \, ds < \infty, \\ \int_0^t |E_s(x)| \, ds &< \infty, \sum_l \int_0^t |F_s^l(x)|^2 \, ds < \infty, \int_0^{t+} \int_{\mathbb{U}_0} |G(s-, x, u)|^2 n(du) \, ds < \infty. \end{aligned}$$

Proposition 5.1. Assume that $\xi_t(\omega, \cdot) \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$, $E_t(\omega, \cdot) \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$, $F_t^l(\omega, \cdot) \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R})$ and $G(t, \omega, \cdot, u) \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R})$ for $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and $u \in \mathbb{U}_0$. Then

$$\begin{aligned} & d\xi_t(\eta_t) \\ &= \left(\frac{\partial \xi_t(\eta_t)}{\partial x^l} \right) e_t^l \, dt + \left(\frac{\partial \xi_t(\eta_t)}{\partial x^l} \right) f_t^{lj} \, dW_t^j + \frac{1}{2} \left(\frac{\partial^2 \xi_t(\eta_t)}{\partial x^l \partial x^i} \right) f_t^{lj} f_t^{ij} \, dt \\ &+ \int_{\mathbb{U}_0} \left[\xi_{t-}(\eta_{t-} + g(t-, u)) - \xi_{t-}(\eta_{t-}) - \left(\frac{\partial \xi_{t-}(\eta_{t-})}{\partial x^l} \right) g^l(t-, u) \right] n(du) \, dt \end{aligned}$$

$$\begin{aligned}
& + E_t(\eta_t) dt + F_t^l(\eta_t) dW_t^l + f_t^{li} \left(\frac{\partial F_t^i(\eta_t)}{\partial x^l} \right) dt \\
& + \int_{\mathbb{U}_0} [\xi_{t-}(\eta_{t-} + g(t-, u)) - \xi_{t-}(\eta_{t-}) + G(t-, \eta_{t-} + g(t-, u), u)] \tilde{N}_q(dt, du) \\
& + \int_{\mathbb{U}_0} [G(t-, \eta_{t-} + g(t-, u), u) - G(t-, \eta_{t-}, u)] n(du) dt.
\end{aligned}$$

The Itô-Ventzell formula for $t < 0$ could be proved by the similar method to one in [18].

Proof of formula (14) in Section 3:

We examine the right hand side of the formula (14), and show that it is equal to the left hand side. By the Fubini theorem [17], we obtain that

$$\begin{aligned}
& \int_s^t \left(\int_s^u e^{r-u} \frac{\partial \sigma_i}{\partial x}(h_r^{x,u}) D(r, u) d\hat{L}_r^i \right) du - \int_s^t \left(\int_s^u e^{r-u} \sigma_i(h_r^{x,u}) d\hat{L}_r^i \right) du \\
& = \int_s^t \left(\int_s^u \frac{\partial}{\partial v} [e^{r-v} \sigma_i(h_r^{x,v})] |_{v=u} d\hat{L}_r^i \right) du \\
& = \int_s^t \left(\int_s^t \frac{\partial}{\partial v} [e^{r-v} \sigma_i(h_r^{x,v})] |_{v=u} I_{[s,u]}(r) d\hat{L}_r^i \right) du \\
& = \int_s^t \left(\int_s^t \frac{\partial}{\partial v} [e^{r-v} \sigma_i(h_r^{x,v})] |_{v=u} I_{[s,u]}(r) du \right) d\hat{L}_r^i \\
& = \int_s^t \left(\int_s^r \frac{\partial}{\partial v} [e^{r-v} \sigma_i(h_r^{x,v})] |_{v=u} I_{[s,u]}(r) du \right) d\hat{L}_r^i \\
& \quad + \int_s^t \left(\int_r^t \frac{\partial}{\partial v} [e^{r-v} \sigma_i(h_r^{x,v})] |_{v=u} I_{[s,u]}(r) du \right) d\hat{L}_r^i \\
& = e^{-t} \int_s^t e^r \sigma_i(h_r^{x,t}) d\hat{L}_r^i - \int_s^t \sigma_i(h_r^{x,r}) d\hat{L}_r^i.
\end{aligned}$$

This proves the formula (14) in Section 3.

REFERENCES

- [1] D. Applebaum: *Lévy Processes and Stochastic Calculus*. Second Edition, Cambridge Univ. Press, Cambridge, 2009.
- [2] L. Arnold: *Random dynamical systems*, Springer-Verlag Berlin, 1998.
- [3] P. Billingsley, *Weak Convergence of Probability Measures*, John Wiley/Sons, New York, Second Edition, 1999.
- [4] E. A. Coayla-Teran, S. E. A. Mohammed and P. R. C. Ruffino: Hartman-Grobman theorem along hyperbolic stationary trajectories, *Discrete and Continuous Dynamical Systems*, 17(2007)281-292.
- [5] F. Flandoli and H. Lisei: Stationary conjugation of flows for parabolic SPDEs with multiplicative noise and some applications, *Stochastic Analysis and Applications*, 22(2005)1385-1420.
- [6] F. Flandoli and B. Schmalfuss: Random attractors for the 3D stochastic Navier Stokes equation with multiplicative noise, *Stochastics and Stochastic Rep.*, 59(1996)21-45.
- [7] T. Fujiwara, H. Kunita: Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group, *J. Math. Kyoto Univ.*, 25(1985)71-106.
- [8] M. Grigoriu: Lyapunov exponents for nonlinear systems with Poisson white noise, *Physics Letters A*, 217(1996)258-262.
- [9] S. He, J. Wang and J. Yan: *Semimartingale Theory and Stochastic Calculus*, Science Press and CRC Press Inc., 1992.

- [10] N. Ikeda, S. Watanabe: *Stochastic differential equations and diffusion processes*, 2nd ed., North-Holland/Kodansha, Amsterdam/Tokyo, 1989.
- [11] P. Imkeller and C. Lederer: The cohomology of stochastic and random differential equations, and local linearization of stochastic flows, *Stochastics and Dynamics*, 2(2002)131-159.
- [12] P. Imkeller and B. Schmalfuss: The conjugacy of stochastic and random differential equations and the existence of global attractors, *Journal of Dynamics and Differential Equations*, 13(2001)215-249.
- [13] G. Kager and M. K. R. Scheutzw: Generation of one-sided random dynamical systems by stochastic differential equations, *Electronic Journal of Probability*, 2(1997)1-17.
- [14] H. Kunita: Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and Stochastic Analysis: New Perspectives*, M. M. Rao (eds.), Birkhäuser Boston, 305-373, 2004.
- [15] C. W. Li and G. L. Blankenship: Almost sure stability of linear stochastic systems with Poisson process coefficients, *SIAM J. Appl. Math.*, 46(1986)875-911.
- [16] S. I. Marcus: Modelling and approximations of stochastic differential equations driven by semimartingales, *Stochastics*, 4(1981)223-245.
- [17] S. Peszat and J. Zabczyk: *Stochastic Partial Differential Equations with Lévy Noise*, Cambridge University Press, 2007.
- [18] H. Qiao: Homeomorphism Flows for Non-Lipschitz SDEs Driven by Lévy Processes, *Acta Mathematica Scientia*, 32(2012)1115-1125.
- [19] H. Qiao and J. Duan: Multiplicative ergodic theorem for discontinuous random dynamical systems, <http://arXiv:1204.5050v2>.
- [20] K. Sato: *Lévy processes and infinitely divisible distributions*, Cambridge University Press, 1999.
- [21] K. R. Schenk-Hoppé: Bifurcation scenarios of the noisy Duffing-van der Pol oscillator, *Nonlin. Dynam.*, 11(1996)255-274.
- [22] M. Spivak: *A Comprehensive Introduction to Differential Geometry*, Vol. 1, 2nd ed., Pub. Berkeley, 1979.
- [23] T. Wanner: Linearization of Random Dynamical Systems. In *Dynamics Report*, Volumn 4. Springer-Verlog, New York, 1995.
- [24] W. A. Woyczynski: Lévy processes in the physical sciences. In *Lévy Processes: Theory and Applications*, O. E. Barndorff-Nielsen, T. Mikosch and S. I. Resnick (Eds.), 241-266, Birkhäuser, Boston, 2001.
- [25] Y. Xie: The random attractors of stochastic Duffing-Van der Pol equations with jumps, *Chinese Journal of Applied Probability and Statistics*, 26(2010)9-23.